Efficient Optimization for Rank-based Loss Functions

Pritish Mohapatra*
IIIT Hyderabad
pritish.mohapatra@research.iiit.ac.in

Michal Rolínek*
IST Austria
michal.rolinek@ist.ac.at

C.V. Jawahar
IIIT Hyderabad
jawahar@iiit.ac.in

Vladimir Kolmogorov
IST Austria
vnk@ist.ac.at

M. Pawan Kumar
University of Oxford
pawan@robots.ox.ac.uk

Abstract

The accuracy of information retrieval systems is often measured using complex non-decomposable loss functions such as the average precision (AP) or the normalized discounted cumulative gain (NDCG). Given a set of positive (relevant) and negative (non-relevant) samples, the parameters of a retrieval system can be estimated using a rank SVM framework, which minimizes a regularized convex upper bound on the empirical loss. However, the high computational complexity of loss-augmented inference, which is required to learn a rank SVM, prohibits its use in large training datasets. To alleviate this deficiency, we present a novel quicksort flavored algorithm for a large class of non-decomposable loss functions. We provide a complete characterization of the loss functions that are amenable to our algorithm. Furthermore, we prove that no comparison based algorithm can improve upon the computational complexity of our approach asymptotically. We demonstrate that it is possible to reduce the constant factors of the complexity by exploiting the special structure of the AP loss. Using the PASCAL VOC action recognition and object detection datasets, we show that our approach provides significantly better results than baseline methods that use a simpler decomposable loss in comparable runtime.

1 Introduction

Information retrieval systems require us to rank a set of samples according to their relevance to a query. The parameters of a retrieval system can be estimated by minimizing the prediction risk on a training dataset, which consists of positive and negative samples. Here, positive samples are those that are relevant to the query, and negative samples are those that are not relevant to the query. Typically, the risk is measured using a user-specified loss function. For information retrieval, several intuitive loss functions have been proposed in the literature. These include simple decomposable losses (that is, loss functions that decompose over each training sample) such as 0-1 loss [1, 2] and the area under the ROC curve [3, 4], as well as the more complex non-decomposable losses (that is, loss functions that depend on the entire training dataset) such as the average precision (AP) [5, 6] and the normalized discounted cumulative gain (NDCG) [7, 8].

Several risk minimization frameworks proposed in the literature are applicable to the problem of learning the parameters of a retrieval system: structured support vector machines (SSVM) [9, 10], neural networks [11], decision forests [12], and boosting [13]. Regardless of which framework is employed, a key practical problem that one needs to address is how to efficiently minimize the empirical risk. Our work addresses this problem in the context of the complex non-decomposable loss functions for information retrieval systems. In particular, we focus on the aforementioned AP and NDCG measures of risk. For clarity, we restrict our description to the SSVM framework, while noting that our work is also applicable to other learning frameworks.

The SSVM framework provides a linear prediction rule to obtain a structured output for an input. In this work, the structured output represents a ranking of a given set of samples. Henceforth, we will refer to this restriction of an SSVM as the rank SVM framework. A rank SVM provides the score of a putative ranking as the dot product of its parameters and the joint feature vector of the input samples and their ranking. The prediction requires us to maximize the score over all possible rankings. During training, the parameters of a rank SVM are estimated by minimizing a regularized convex upper bound on the prediction loss over the training dataset. While in theory the rank SVM framework can be employed in conjunction with any loss function, in practice its feasibility depends on the computational efficiency of the corresponding *loss-augmented inference*. In other words, given the current estimate of the parameters, it is important to be able to efficiently maximize the sum of the score and the loss function over all possible rankings.

When the loss function is decomposable, the loss-augmented inference problem can be solved efficiently by

*The first two authors contributed equally.

independently considering each training sample. However, for non-decomposable loss functions, it presents a hard computational challenge. Specifically, given a training dataset with P positive and N negative samples, the best known algorithms for loss-augmented inference for AP and NDCG loss functions have a complexity of $O(PN + N \log N)$ [8, 6]. Since the number of negative samples N can be very large in practice, this prohibits their use on large datasets.

In order to alleviate the computational deficiencies of the AP and the NDCG loss functions, we make four contributions. First, we characterize a large class of ranking based loss functions that are amenable to a novel quicksort flavored optimization algorithm for the corresponding loss-augmented inference problem. We refer to the class of loss functions as *QS-suitable*. Second, we show that the AP and the NDCG loss functions are QS-suitable, which allows us to reduce the complexity of the corresponding loss-augmented inference to $O(N \log P)$. Third, we prove that there cannot exist a comparison based method for loss-augmented inference that can provide a better asymptotic complexity than our quicksort flavored approach. Fourth, we exploit the special structure of AP to further speed up the corresponding loss-augmented inference problem. Note that the improvement is only in terms of a constant factor, as an asymptotic improvement is not possible. Nonetheless, it is very effective in practice to reduce the time required for each iteration of our quicksort flavored algorithm.

We demonstrate the efficacy of our approach on two challenging problems—action recognition and object detection—using publicly available datasets. Rather surprisingly, we show that the optimization of the complex non-decomposable AP and NDCG loss functions can be carried out faster than that of the simple decomposable 0-1 loss. Specifically, while each loss-augmented inference call is more expensive for AP and NDCG loss functions, it takes fewer calls in practice to estimate the parameters of the corresponding rank SVM.

2 Background

We begin by providing a brief description of the general SVM framework that employs a rank-based loss function, hereby referred to as the rank SVM. Note that this framework is the same as the one employed in previous work [8, 14, 15, 6]. The two specific instantiations of the rank SVM framework that are of interest to us employ the average precision (AP) loss and the normalized discounted cumulative gain (NDCG) loss respectively. A detailed description of the two aforementioned loss functions is provided in the subsequent subsection.

2.1 The Rank SVM Framework

Input. The input of a rank SVM is a set of n samples, which we denote by $\mathbf{X} = \{\mathbf{x}_i, i = 1, \ldots, n\}$. For example, each sample can represent an image and a bounding box of a person present in the image. In addition, we are also provided with a query, which in our example could represent an action such as 'jumping'. Each sample can either belong to the positive class (that is, the sample is relevant to the query) or the negative class (that is, the sample is not relevant to the query). For example, if the query represents the action 'jumping' then a sample is positive if the corresponding person is performing the jumping action, and negative otherwise.

The positive and the negative samples are denoted by $\mathcal{P}$ and $\mathcal{N}$ respectively. In other words, if $\mathbf{x} \in \mathcal{P}$ and $\mathbf{y} \in \mathcal{N}$ then $\mathbf{x}$ belongs to the positive class and $\mathbf{y}$ belongs to the negative class. Thoroughout the paper, we assume that the sets $\mathcal{P}$ and $\mathcal{N}$ are provided during training, but are not known during testing.

Output. Given a query and a set of n samples $\mathbf{X}$, the desired output of the framework is a ranking of the samples according to their relevance to the query. This is often represented by an $n \times n$ ranking matrix $\mathbf{R}$ defined as follows:

- $\mathbf{R}_{\mathbf{x},\mathbf{y}} = 1$ if $\mathbf{x}$ is ranked higher than $\mathbf{y}$;
- $\mathbf{R}_{\mathbf{x},\mathbf{y}} = -1$ if $\mathbf{x}$ is ranked lower than $\mathbf{y}$;
- $\mathbf{R}_{\mathbf{x},\mathbf{y}} = 0$ if $\mathbf{x}$ and $\mathbf{y}$ are ranked the same.

In other words, the matrix $\mathbf{R}$ is anti-symmetric and its elements, which belong to the set $\{-1, 0, 1\}$, represent the relative ranking of a pair of samples.

Given the sets $\mathcal{P}$ and $\mathcal{N}$ during training, we construct a ground truth ranking matrix $\mathbf{R}^*$, which ranks each positive sample above all the negative samples. Formally, the ground truth ranking matrix $\mathbf{R}^*$ is defined as follows:

- $\mathbf{R}^*_{\mathbf{x},\mathbf{y}} = 1$ if $\mathbf{x} \in \mathcal{P}$ and $\mathbf{y} \in \mathcal{N}$;
- $\mathbf{R}^*_{\mathbf{x},\mathbf{y}} = -1$ if $\mathbf{x} \in \mathcal{N}$ and $\mathbf{y} \in \mathcal{P}$;
- $\mathbf{R}^*_{\mathbf{x},\mathbf{y}} = 0$ if $\mathbf{x}, \mathbf{y} \in \mathcal{P}$ or $\mathbf{x}, \mathbf{y} \in \mathcal{N}$.

Note that the ground truth ranking matrix only defines a partial ordering on the samples since $\mathbf{R}^*_{i,j} = 0$ for all pairs of positive and negative samples. We will refer to rankings where no two samples are ranked equally as *proper rankings*. Without loss of generality, we will treat all rankings other than the ground truth one as a proper ranking by breaking ties arbitrarily.

Features and Joint Feature Vectors. We denote the feature vector of a sample $\mathbf{x}$ by $\psi(\mathbf{x})$. For example, if $\mathbf{x}$ denotes the bounding box of a person in an image, then $\psi(\mathbf{x})$ could represent its Poselet feature [16]. We do not place any restrictions on the form of the feature vector. Given a set

of samples $\mathbf{X}$ and a ranking $\mathbf{R}$, we define the joint feature vector as

$$\Psi(\mathbf{X}, \mathbf{R}) = \frac{1}{|\mathcal{P}|\,|N|} \sum_{\mathbf{x}\in\mathcal{P}} \sum_{\mathbf{y}\in\mathcal{N}} \mathbf{R}_{\mathbf{x},\mathbf{y}}(\psi(\mathbf{x}) - \psi(\mathbf{y})). \quad (1)$$

In other words, the joint feature vector is the scaled sum of the difference between the features of all pairs of samples, where the scaling is specified by the ranking. Note that, for any ranking $\mathbf{R}$, the size of the joint feature vector is the same as the size of the feature vector of a sample.

Parameters and Prediction. The parameter vector of rank SVM, denoted by $\mathbf{w}$, provides a linear scoring function for any ranking $\mathbf{R}$ of an input. Specifically, the score of the ranking is the dot product of the parameters and the joint feature vector of the input and the ranking. Given the parameters $\mathbf{w}$, the ranking of an input $\mathbf{X}$ is predicted by maximizing the score, that is, by solving the following optimization problem:

$$\mathbf{R}(\mathbf{w}) = \operatorname*{argmax}_{\mathbf{R}} \mathbf{w}^\top \Psi(\mathbf{X}, \mathbf{R}). \quad (2)$$

The special form of the joint feature vector in equation (1) enables us to efficiently obtain the predicted ranking $\mathbf{R}(\mathbf{w})$ by sorting the samples in descending order of their individual scores. Here, the individual score of a sample $\mathbf{x}$ is defined as

$$s(\mathbf{x}) = \mathbf{w}^\top \psi(\mathbf{x}). \quad (3)$$

We refer the reader to [14, 6] for details.

Parameter Estimation. We now turn our attention to the problem of estimating the parameters of a rank SVM given the input samples $\mathbf{X}$, together with their classification into positive and negative sets $\mathcal{P}$ and $\mathcal{N}$ respectively. To this end, we will minimize a regularized upper bound on the empirical risk of prediction. The empirical risk is computed using a user-specified loss function $\Delta(\mathbf{R}^*, \mathbf{R}(\mathbf{w}))$ where $\mathbf{R}^*$ is the ground truth ranking that is determined by $\mathcal{P}$ and $\mathcal{N}$ and $\mathbf{R}(\mathbf{w})$ is the predicted ranking as shown in equation (2). For now, we will assume that the loss function is general. In the next subsection, we will define the two widely used loss functions—AP loss and NDCG loss—that are the focus of this work.

Formally, the parameters of a rank SVM are obtained by solving the following convex problem:

$$\begin{aligned} \min_{\mathbf{w},\xi} \quad & \frac{1}{2}\|\mathbf{w}\| + C\xi, \quad (4) \\ \text{s.t.} \quad & \mathbf{w}^T\Psi(\mathbf{X}, \mathbf{R}^*) - \mathbf{w}^T\Psi(\mathbf{X}, \mathbf{R}) \geq \Delta(\mathbf{R}^*, \mathbf{R}) - \xi, \forall \mathbf{R}. \end{aligned}$$

Optimization for Parameter Estimation. At first glance, problem (4) appears to be computationally intractable due to the large number of constraints. However, it has been shown to lend itself to several efficient algorithms [17, 18, 19]. The common requirement of all the aforementioned methods is to be able to efficiently solve the problem of *loss-augmented inference*. In other words, the key to learning a rank SVM is to solve the following problem for any given parameter vector $\mathbf{w}$ and input $\mathbf{X}$:

$$\bar{\mathbf{R}} = \operatorname*{argmax}_{\hat{\mathbf{R}}} \mathbf{w}^\top \Psi(\mathbf{X}, \hat{\mathbf{R}}) + \Delta(\mathbf{R}^*, \hat{\mathbf{R}}). \quad (5)$$

The solution to the above problem allows us to compute the term

$$\Psi(\mathbf{X}, \bar{\mathbf{R}}) - \Psi(\mathbf{X}, \mathbf{R}^*),$$

which has various useful interpretations: the subgradient of the unconstrained version of problem (4), which enables subgradient descent [18]; the conditional gradient of the dual of problem (4), which enables Frank-Wolfe optimization [19]; or the most violated constraint, which enables the efficient cutting plane algorithm of [17] that is used in our experiments.

2.2 Loss Functions

We now describe two specific instantiations of the general rank SVM framework that will be used throughout the remainder of the paper. The two instantiations arise due to the choice of the loss functions. The first instantiation uses the average precision (AP) loss, which is very popular in the computer vision community as evidenced by its use in the various challenges of PASCAL VOC [20]. The second instantiation uses the normalized discounted cumulative gain (NDCG) loss, which is very popular in the information retrieval community [8].

Proper Loss Functions. Before we describe the two loss functions in detail, we note that all the rankings $\mathbf{R}$ other than the ground truth ranking $\mathbf{R}^*$ will be assumed to be proper for the remainder of the paper. Recall that a proper ranking is one that does not assign an equal rank to any pair of samples. We will refer to a loss function defined between a ground truth ranking $\mathbf{R}^*$ and a proper ranking $\mathbf{R}$ as a proper loss function. Henceforth, we will restrict ourselves to only proper rankings when dealing with problem (5).

Notation. In order to specify the loss functions, and our efficient algorithms for problem (5), it would be helpful to introduce some additional notation. We define $ind(\mathbf{x})$ to be the index of a sample $\mathbf{x}$ in $\mathbf{R}$. Note that the notation does not explicitly depend on $\mathbf{R}$ as the ranking will always be clear from context. If $\mathbf{x} \in \mathcal{P}$ (that is, for a positive sample), we define $ind^+(\mathbf{x})$ as the index of $\mathbf{x}$ in the total order of positive samples induced by $\mathbf{R}$. For example, if $\mathbf{x}$ is the highest ranked positive sample then $ind^+(\mathbf{x}) = 1$ even though $ind(\mathbf{x})$ need not necessarily be 1 (in the case where some negative samples are ranked higher than $\mathbf{x}$). For a negative sample $\mathbf{x} \in \mathcal{N}$, we define $ind^-(\mathbf{x})$ analogously: $ind^-(\mathbf{x})$ is the index of $\mathbf{x}$ in the total order of negative samples induced by $\mathbf{R}$.

AP Loss. Using the above notation, we can now concisely define the average precision (AP) loss of a proper ranking

$\mathbf{R}$ given the ground truth ranking $\mathbf{R}^*$ as follows:

$$\Delta_{AP}(\mathbf{R}^*, \mathbf{R}) = 1 - \frac{1}{|\mathcal{P}|} \sum_{\mathbf{x} \in \mathcal{P}} \frac{ind^+(\mathbf{x})}{ind(\mathbf{x})}.$$

For example, consider an input $\mathbf{X} = \{\mathbf{x}_1, \cdots, \mathbf{x}_8\}$ where $\mathbf{x}_i \in \mathcal{P}$ for $1 \leq i \leq 4$, and $\mathbf{x}_i \in \mathcal{N}$ for $5 \leq i \leq 8$. In other words, the first four samples are positive while the last four samples are negative. If the proper ranking $\mathbf{R}$ induces the order

$$(\mathbf{x}_1, \mathbf{x}_3, \mathbf{x}_8, \mathbf{x}_4, \mathbf{x}_5, \mathbf{x}_2, \mathbf{x}_6, \mathbf{x}_7), \tag{6}$$

then

$$\Delta_{AP}(\hat{\mathbf{R}}, \mathbf{R}) = 1 - \frac{1}{4}\left(\frac{1}{1} + \frac{2}{2} + \frac{3}{4} + \frac{4}{6}\right) \approx 0.146.$$

NDCG Loss. We define a discount $D(i) = 1/\log_2(1+i)$ for all $i = 1, \cdots, |\mathcal{P}|$. This allows us to obtain a loss function based on the normalized discounted cumulative gain as

$$\Delta_{NDCG}(\mathbf{R}^*, \mathbf{R}) = 1 - \frac{\sum_{\mathbf{x} \in \mathcal{P}} D(ind(\mathbf{x}))}{\sum_{i=1}^{|\mathcal{P}|} D(i)}.$$

For example, consider the aforementioned input where the first four samples are positive and the last four samples are negative. For the ranking $\mathbf{R}$ that induces the order (6), we can compute

$$\Delta_{NDCG}(\hat{\mathbf{R}}, \mathbf{R}) = 1 - \frac{1 + \log_2^{-1} 3 + \log_2^{-1} 5 + \log_2^{-1} 7}{1 + \log_2^{-1} 3 + \log_2^{-1} 4 + \log_2^{-1} 5} \approx 0.056.$$

2.3 State of the Art

The main focus of our work is to enable efficient optimization of problem (5) for the AP loss and the NDCG loss. This in turn will allow us to efficiently learn the parameters of a rank SVM for a given training data by solving problem (4). Before we describe our optimization approach in detail, we present a brief overview of the state of the art for solving problem (5) with the aim of contextualizing our contributions. We focus on two aspects of prior work, namely their correctness and their computational complexity.

Correctness. Chakrabarti *et al.* [8] use a slightly modified definition of the discount $D(\cdot)$ as

$$D(i) = \begin{cases} 1 & 1 \leq i \leq 2 \\ 1/\log_2(i) & i > 2 \end{cases}.$$

For the resulting NDCG loss, they propose a greedy algorithm for solving problem (5). However, with the above definition of a discount, it is possible to obtain a corner-case where their proof of correctness of the greedy algorithm is not valid (specifically, there exists a counter-example for fact 3.4 of [8]). In order for the greedy algorithm to be correct, it turns out that the convexity of $D(i)$ is essential. As our main concern is to improve the efficiency of learning a rank SVM for various loss functions, we defer the discussion on the counter-example for the previous discount function and its corrected version to the appendices.

Computational Complexity. The previous best algorithms for solving problem (5) had runtimes $O(|\mathcal{N}|\,|\mathcal{P}| + |\mathcal{N}| \log |\mathcal{N}|)$ for both Δ_{AP} [6] and Δ_{NDCG} [8]. In the case of Δ_{NDCG} the authors also suggest to use a cut-off k in the definition of discount $D(i)$, setting $D(i) = 0$ for $i \geq k$. With this simplification they achieved a reduced complexity of $O((|\mathcal{N}| + |\mathcal{P}|)\log(|\mathcal{P}| + |\mathcal{N}|) + k^2)$.

The preliminary version of our work describes two algorithms for Δ_{AP} [15]. The first algorithm leads to a complexity of $O(|\mathcal{P}|^2 + |\mathcal{N}| \log |\mathcal{N}|)$. The second is a heuristic that improves the efficiency in practice, but does not result in any guaranteed improvements in the computational complexity.

3 Our Contributions

We present a novel quicksort flavored algorithm for problem (5), which achieves a complexity of $O(|\mathcal{N}| \log |\mathcal{P}|)$ for a large class of ranking based loss functions. Note that the quadratic terms in the complexity vanish. Our only assumption is that $|\mathcal{N}| > |\mathcal{P}|$, which often holds in practice as it is easier to collect negative samples than positive ones.

We provide a complete characterization of the loss functions that are suitable for our algorithm, which we refer to as the *QS-suitable* class of loss functions. We show that both the AP loss and the NDCG loss are QS-suitable, which enables the use of our efficient algorithm to learn the parameters of the corresponding rank SVM.

Furthermore, we also show that our runtime also meets the optimal complexity among comparison-based algorithms for problem (5). This rules out the possibility of improving the asymptotic computational complexity of solving problem (5) compared to our algorithm. However, it may still be possible to improve the underlying constants within the complexity, thereby speeding up the algorithm. Indeed, we will shortly demonstrate that the additional structure of the AP loss allows us to incorporate further improvements.

4 Quicksort Flavored Optimization

Given the parameters $\mathbf{w}$ of a rank SVM, as well as a set of samples $\mathbf{X}$, we are interested in obtaining the most violated ranking by solving problem (5). At first glance, the problem seems to require us to obtain a ranking matrix $\bar{\mathbf{R}}$. However, it turns out that we do not explicitly require a ranking matrix. Instead, we need to compute two related quantities to enable optimization via any of the standard methods for estimating the parameters of an SVM. First, the joint

feature vector of the most violated ranking $\Psi(\mathbf{X}, \bar{\mathbf{R}})$. In order to compute $\Psi(\mathbf{X}, \bar{\mathbf{R}})$ without explicitly computing the matrix $\bar{\mathbf{R}}$, we observe that the joint feature vector is a linear combination of the sample feature vectors, as shown in equation (1). We denote the vector of coefficients of this linear combination by $\mathbf{c}$, and design an algorithm to directly compute $\mathbf{c}$. Note that $\Psi(\mathbf{X}, \bar{\mathbf{R}})$ can be computed from $\mathbf{c}$ in $O(|\mathcal{N}| + |\mathcal{P}|)$ time. Second, we require the value of the loss function $\Delta(\mathbf{R}^*, \bar{\mathbf{R}})$ at the most violated ranking. Once again, our algorithm aims to accomplish this without explicitly computing the ranking matrix.

In more detail, our algorithm uses an intermediate representation of the ranking using the notion of interleaving ranks. Given a ranking $\mathbf{R}$ and a positive sample $\mathbf{x}$, the interleaving rank of $\mathbf{x}$, denoted by $rank(\mathbf{x})$, is defined as one plus the number of negative samples preceding $\mathbf{x}$ in $\mathbf{R}$. Analogously, for a negative sample $\mathbf{x}$, the interleaving rank $rank(\mathbf{x})$ is defined as one plus the number of positive samples preceding $\mathbf{x}$ in $\mathbf{R}$. Note that, similar to our notation for $ind(\cdot)$, $ind^+(\cdot)$ and $ind^-(\cdot)$, we have dropped the dependency of $rank(\cdot)$ on $\mathbf{R}$ as the ranking matrix would be clear from context. Note that the interleaving rank of all the samples does not specify the total ordering of all the samples according to $\mathbf{R}$ as it ignores the relative ranking of the positive samples among themselves, and the relative ranking of the negative samples among themselves. However, as will be seen shortly, for a large class of ranking based loss functions, interleaving ranks are sufficient to compute both the joint feature vector and the loss of a given ranking function.

The rest of the section is organized as follows. We begin by defining the class of loss functions that are amenable to a quicksort flavored algorithm, which we call QS-suitable loss functions. Subsection 4.2 provides some key observations that exploit the properties of QS-suitable loss functions to aid the development of an efficient algorithm for problem (5). Subsection 4.3 describes our quicksort flavored divide-and-conquer approach in detail. Finally, subsection 4.4 provides the computational complexity of our approach. Furthermore, it establishes that there cannot exist any other comparison based algorithm that has a better asymptotic computational complexity compared to our approach.

4.1 QS-Suitable Loss Functions

A proper loss function $\Delta = \Delta(\mathbf{R}^*, \mathbf{R})$ is called *QS-suitable* if it meets the following four conditions.

(C1) **±-pattern dependence.** Given a proper ranking $\mathbf{R}$, its ±-pattern is defined as the pattern obtained by replacing each positive sample with a "+" symbol and each negative sample with a "−" symbol. Our first condition states that, for a proper ranking $\mathbf{R}$, the value of $\Delta(\mathbf{R}^*, \mathbf{R})$ depends only on the ±-pattern of $\mathbf{R}$ and can be computed from this pattern in $O(|\mathcal{N}|+|\mathcal{P}|)$ time.

(C2) **Additive decomposability with respect to negative samples.** There are functions $\delta_j \colon \{1, \ldots, |\mathcal{P}| + 1\} \to \mathbb{R}$ for $j = 1, \ldots, |\mathcal{N}|$ such that for a proper ranking $\mathbf{R}$ one can write

$$\Delta(\mathbf{R}^*, \mathbf{R}) = \sum_{\mathbf{x} \in \mathcal{N}} \delta_{ind^-(\mathbf{x})}(rank(\mathbf{x})).$$

(C3) **j-monotonicity of discrete derivative.** For every $1 \leq j < |\mathcal{N}|$ and $1 \leq i \leq |\mathcal{P}|$ we have

$$\delta_{j+1}(i+1) - \delta_{j+1}(i) \geq \delta_j(i+1) - \delta_j(i).$$

(C4) **Fast evaluation of discrete derivative.** For any $j \in \{1, \ldots, |\mathcal{N}|\}$ and $i \in \{1, \ldots, |\mathcal{P}|\}$, can the value $\delta_j(i+1) - \delta_j(i)$ be computed in constant time.

While the above conditions may at first appear to be restrictive, the following proposition establishes their usefulness for our work.

Proposition 1 *Both Δ_{AP} and Δ_{NDCG} are QS-suitable.*

The proof of the above proposition is provided in Appendix B. Having established that both the AP and the NDCG loss are QS-suitable, the rest of the section will deal with a general QS-suitable loss function. A reader who is interested in employing another loss function need only check whether the above four conditions are satisfied in order to use our approach.

4.2 Key Observations for QS-Suitable Loss

Before describing our algorithm in detail, we first provide a set of key observations which enable efficient optimization for QS-suitable loss functions. To this end, it would be useful to define two arrays, one for the positive sample scores and one for the negative sample scores. Specifically, given a parameter vector $\mathbf{w}$, the score for a sample can be computed as shown in equation (3). During training, the samples are partitioned into the positive class and the negative class. This allows us to define an array $\{s_i^+\}_{i=1}^{|\mathcal{P}|}$ of positive sample scores and an array $\{s_i^-\}_{i=1}^{|\mathcal{N}|}$ of negative sample scores. Without loss of generality, we assume that the elements of the two aforementioned arrays are distinct.[1]

Using the above notation, we will describe four key observations regarding QS-suitable loss functions. Their proofs are for the most part straightforward generalizations of results that appeared in [6] and [15] in the context of the AP loss Δ_{AP} and can be found in Appendix A.

[1] In a general case we work with scores $\hat{s}_i^+ = s_i^+ + i \cdot \epsilon$ for some $\epsilon > 0$. If ϵ is sufficiently small, then an optimal solution of scores $\{\hat{s}_i^+\}$ will also be an optimal solution for the original scores. Algorithmically, this means that when we need to comparing scores s_i^+ and s_j^+ we check whether $i < j$ in case when $s_i^+ = s_j^+$. The same argument applies for the array of negative sample scores.

Observation 1 *There exists an optimal solution $\hat{\mathbf{R}}$ of problem (5) in which the positive samples appear in the descending order of their scores s_i^+ and also the negative samples appear in descending order of their scores s_i^-.*

Now it would seem natural to sort the arrays $\{s_i^+\}$ and $\{s_i^-\}$ in descending order. However, we are aiming for complexity below $O(|\mathcal{N}|\log|\mathcal{N}|)$, therefore we can not afford to sort the negative scores. On the other hand, since $|\mathcal{P}| < |\mathcal{N}|$, we are allowed to sort the array of positive scores $\{s_i^+\}$. For purely notational purposes let $\{s_i^*\}$ be the array $\{s_i^-\}$ sorted in descending order. Furthermore, for $j \in \{1,\dots|\mathcal{N}|\}$ we denote the index of s_j^- in $\{s_i^*\}$ as j^*. It is worth keeping in mind that, for the purposes of the algorithm, we do not have access to the array $\{s_i^*\}$. However, we will compute j^* for many values of $j \in \{1,\dots,|\mathcal{N}|\}$. Note that in the sought ranking $\hat{\mathbf{R}}$ if the score of $\mathbf{x} \in \mathcal{N}$ is s_j^-, then $j^* = ind^-(\mathbf{x})$.

Let $\mathcal{R}$ be the set of vectors $\mathbf{r} = (r_1,\dots,r_{|\mathcal{N}|})$, where for each $1 \le j \le |\mathcal{N}|$, one has $1 \le r_j \le |\mathcal{P}|+1$, and $r_k \le r_l$ whenever $k < l$. To each vector $\mathbf{r} \in \mathcal{R}$ we can associate the unique ranking $\mathbf{R_r}$ satisfying the condition from Observation 1 such that $r_{j^*} = rank(\mathbf{x})$ where $\mathbf{x} \in \mathcal{N}$ is the sample with score s_j^-. Then it suffices to restrict the optimization problem (5) to rankings that correspond to vectors $\mathbf{r} \in \mathcal{R}$.

In the following observation we uncover that we can compute the output given the vector $\mathbf{r}$ and a weaker version of the array $\{s_j^*\}$.

Observation 2 *Assume we are given a vector $\mathbf{r} \in \mathcal{R}$ and an array $\{s_j^{**}\}$ which is a rearrangement of the array $\{s_j^-\}$ in which $s_i^{**} > s_j^{**}$ whenever $r_i < r_j$. Then we can compute the entire output of the algorithm, that is the vector $\mathbf{c}$ as well as the value $\Delta(\mathbf{R}^*, \mathbf{R_r})$ in time $O(|\mathcal{N}| + |\mathcal{P}|)$. Moreover, the mapping that produces vector $\mathbf{c}$ given the vector $\mathbf{r}$ is injective.*

Note that $\{s_j^*\}$ would meet the condition for $\{s_j^{**}\}$. However, in $\{s_j^{**}\}$ we do not insist on correctly ordering the negative samples with the same interleaving rank. Since typically $|\mathcal{P}| < |\mathcal{N}|$, some interleaving ranks will be shared by many negative samples and therefore $\{s_j^{**}\}$ should be computationally less expensive to produce than s_j^*.

A different phrasing of this observation was presented in the context of a heuristic argument given in the preliminary version of this work [15]. It was experimentally verified that after only few iterations of the SVM optimization procedure most of the negative samples get ranked after all the positive samples. In light of the observation, this could, if detected, potentially save substantial computational effort. Here we exploit this idea even further.

It turns out that not only the loss function is decomposable over the negative samples as given in the condition (C2) but in fact the same holds for the entire objective function (5).

Observation 3 *There are functions $f_j\colon \{1,\dots,|\mathcal{P}|+1\} \to \mathbb{R}$ for $j = 1,\dots,|\mathcal{N}|$ such that for a proper ranking $\mathbf{R}$ which corresponds to some vector $\mathbf{r} \in \mathcal{R}$ the objective function (5) can be written as*

$$\sum_{j=1}^{|\mathcal{N}|} f_j(r_j),$$

where the functions f_j inherit property (C3) and provided the value of s_j^* also the property (C4). In particular, given s_j^* we can compute all the values of $f_j(i)$ for $l \le i \le r$ in $O(r-l)$ time.

This gives us the opportunity to compute the interleaving ranks independently. Let $opt_j = \operatorname{argmax} f_j$ be the optimal interleaving rank for $j = 1,\dots,|\mathcal{N}|$ (if there are multiple such ranks, we pick the highest one, and denote it opt_j). Finally, we set $\mathbf{opt} = \{opt_j\}$.

It is not obvious that greedily maximizing the contribution of each sample greedily would produce a valid vector $opt \in \mathcal{R}$. However, for QS-suitable loss functions, this can indeed be shown to be the case.

Observation 4 *If $i < j$, then $opt_i \le opt_j$. In other words $\mathbf{opt} \in \mathcal{R}$.*

This has three consequences, the last two of which will be incorporated into our algorithm.

- If we had access to $\{s_j^*\}$ we could simply compute opt_j from $j = 1$ to $j = |\mathcal{N}|$. As we have access to the discrete derivative of f_j, each opt_j is determined in $O(|\mathcal{P}|)$ time. This is exactly the $O(|\mathcal{P}|\,|\mathcal{N}| + |\mathcal{N}|\log|\mathcal{N}|)$ algorithm from [6].
- Even without access to $\{s_j^*\}$, we can for fixed j find s_j^*, the j-highest element in $\{s_i^-\}$, in $O(|\mathcal{N}|)$ time. This would too lead to a $O(|\mathcal{P}|\,|\mathcal{N}|)$ algorithm but we may at each step modify $\{s_i^-\}$ slowly introducing the correct order. This will make the future searches for s_j^* more efficient.
- Knowing that $opt_i = opt_j$ for some $i < j$, we can also conclude that $opt_i = opt_k = opt_j$ for each $i < k < j$. This provides a cheap way to compute some parts of the vector $\mathbf{opt}$.

4.3 Divide and Conquer

Algorithms 1 and 2 describe the main steps of our approach. Briefly, we begin by detecting $s^*_{|\mathcal{N}|/2}$ that is median score among the negative samples. We use this to compute $opt_{|\mathcal{N}|/2}$. Given $opt_{|\mathcal{N}|/2}$, we know that for all $j < |\mathcal{N}|/2$ $opt_j \in [1, opt_{|\mathcal{N}|/2}]$ and for all $j > |\mathcal{N}|/2$ $opt_j \in [opt_{|\mathcal{N}|/2}, |\mathcal{P}|+1]$. This observation allows us to employ a divide-and-conquer recursive approach.

In more detail, we use two classical linear time array manipulating procedures Median and Select. The first one outputs the index of the median element. The second one takes as its input an index of a particular element x. It

rearranges the array such that x separates higher-ranked elements from lower-ranked elements (in some total order). For example, if array s^- contains six scores [a b 4.5 6 1 c] then `Median`(3, 5) would return 3 (the index of score 4.5), while calling `Select`(3, 3, 5) would rearrange the array to [a b 1 4.5 6 c] and return 4 (the new index of 4.5). The SELECT procedure is a subroutine of the classical QUICKSORT algorithm.

Using the two aforementioned procedures in conjunction with the divide-and-conquer strategy allows us to compute the entire vector **opt**. Furthermore, the array $\{s^-\}$ acquires the property required for array $\{s_i^{**}\}$. This allows us to compute the entire output by Observation 2.

Algorithm 1: Algorithm for finding the most violated constraint

Input: Unsorted arrays $s^+[1\dots|\mathcal{P}|]$ and $s^-[1\dots|\mathcal{N}|]$ of positive and negative sample scores.

1 `Sort`(s^+, 1, $|\mathcal{P}|$) ▷ *by scores in decreasing order, this takes $O(|\mathcal{P}|\log|\mathcal{P}|)$ time*
2 `OptRanks`(1,$|\mathcal{N}|$, 1, $|\mathcal{P}|+1$)
3 Compute the output, that is $\mathbf{c}$ and $\Delta(\mathbf{R}^*,\mathbf{R})$.
▷ *by Observation 2, this takes $O(|\mathcal{N}|+|\mathcal{P}|)$ time*

Algorithm 2: Recursive procedure for finding all interleaving ranks.

Description: The function finds optimal interleaving rank for all $i\in[\ell^-,r^-]$ given that
(i) array s^- is partially sorted, namely
$\texttt{MAX}(s^-[1\dots\ell^-1])\le\texttt{MIN}(s^-[\ell^-\dots r^-])$ and
$\texttt{MAX}(s^-[\ell^-\dots r^-])\le\texttt{MIN}(s^-[r^-+1\dots|\mathcal{N}|])$;
(ii) optimal interleaving ranks for $i\in[\ell^-,r^-]$ lie in the interval $[\ell^+,r^+]$.

1 **function** `OptRanks`(int ℓ^-, int r^-, int ℓ^+, int r^+)
2 **if** $\ell^+=r^+$ **then**
3 set $opt_i=\ell^+$ for each $i\in[\ell^-,r^-]$ and return
4 **end**
5 $m=$ `Median`(ℓ^-, r^-) ▷ *gives the index of the median score in a subarray of s^-*
6 $m=$ `Select`(m, ℓ^-, r^-) ▷ *splits the subarray by $s=s^-[m]$, returns the new index of s*
7 Find opt_m by trying all options in $[\ell^+,r^+]$
8 **if** $\ell^-<m$ **then** `OptRanks`(ℓ^-, $m-1$, ℓ^+, opt_m)
9 **if** $m<r^-$ **then** `OptRanks`($m+1$, r^-, opt_m, r^+)
10

4.4 Computational Complexity

While the previous subsection provides an elegant divide-and-conquer strategy to estimate the output of problem (5), we are yet to establish its computational efficiency. To this end, we now present the following theorem.

Theorem 2 *If Δ is QS-suitable, then the task (5) can be solved in time $O(|\mathcal{N}|\log|\mathcal{P}|+|\mathcal{P}|\log|\mathcal{P}|+|\mathcal{P}|\log|\mathcal{N}|)$, which in the most common case $|\mathcal{N}|>|\mathcal{P}|$ reduces to $O(|\mathcal{N}|\log|\mathcal{P}|)$.*

The complexity stated in Theorem 2 is in fact achieved by Algorithm 1. In order to prove this, we first need to establish the complexity of Algorithm 2 as $O(|\mathcal{N}|\log|\mathcal{P}|+|\mathcal{P}|+\log|\mathcal{N}|)$. Since the remaining parts of Algorithm 1 take $O(|\mathcal{P}|\log|\mathcal{P}|+|\mathcal{N}|)$ running time, the total complexity of Algorithm 1 will then be $O(|\mathcal{N}|\log|\mathcal{P}|+|\mathcal{P}|\log|\mathcal{P}|+|\mathcal{P}|\log|\mathcal{N}|)$ as claimed.

For the purposes of the running time analysis of Algorithm 2, let us denote $n=r^--\ell^-+1$ and $p=r^+-\ell^++1$, and set $T_{neg}(n,p)$, $T_{pos}(n,p)$ as the total time spent traversing the arrays of negative and positive sample scores, respectively, including recursive calls. The negative score array is traversed in the MEDIAN and SELECT procedures and the positive scores are traversed when searching for opt_m.

Proposition 3 *The runtimes $T_{neg}(n,p)$ and $T_{pos}(n,p)$ satisfy the following recursive inequalities*

$$\begin{aligned}
T_{neg}(n,p)&\le Cn+T_{neg}(n/2,p_1)+T_{neg}(n/2,p_2)\\
&\qquad\text{for some}\quad p_1+p_2=p+1,\\
T_{pos}(n,p)&\le Cp+T_{pos}(n/2,p_1)+T_{pos}(n/2,p_2)\\
&\qquad\text{for some}\quad p_1+p_2=p+1,\\
T_{neg}(n,1)&\le Cn,\qquad T_{neg}(1,p)=0,\\
T_{pos}(n,1)&=0,\qquad T_{pos}(1,p)\le Cp
\end{aligned}$$

for a suitable constant C. These inequalities imply $T_{neg}(n,p)\le C'n\log(1+p)$ and $T_{pos}(n,p)\le C'(p-1)\log(1+n)$ for another constant C'. Thus the running time of Algorithm 2, where $p=|\mathcal{P}|+1$, $n=|\mathcal{N}|$, is $O(|\mathcal{N}|\log|\mathcal{P}|+|\mathcal{P}|\log|\mathcal{N}|)$.

Proof. In both cases we proceed by induction. For the first inequality the base step is trivial for high enough constant C' and for the inductive step we may write

$$\begin{aligned}
T_{neg}(n,p)&\le Cn+T_{neg}(n/2,p_1)+T_{neg}(n/2,p_2)\\
&\le Cn+\frac{1}{2}C'n\log(1+p_1)+\frac{1}{2}C'n\log(1+p_2)\\
&=C'n\left(\frac{C}{C'}+\log\sqrt{(1+p_1)(1+p_2)}\right)\\
&\le C'n\log(p_1+p_2)=C'n\log(1+p)
\end{aligned}$$

where in the last inequality we used that

$$1+(1+p_1)(1+p_2)\le(p_1+p_2)^2$$

for integers p_1, p_2 with $p_1+p_2=p+1\ge3$. That makes the last inequality true for sufficiently high C' (not depending on n and p!).

The proof of the second inequality is an easier variation on the previous technique. □ In principle one can reverse the role of positive and negative samples in the definition of QS-suitable loss functions. Then a statement analogous to Theorem 2 can be derived leading to (suboptimal) complexity $O(|\mathcal{P}|\log|\mathcal{N}| + |\mathcal{N}|\log|\mathcal{N}| + |\mathcal{N}|\log|\mathcal{P}|)$.

4.5 Lower Bound on Complexity

A natural question that arises is whether the divide-and-conquer strategy outlined in subsection 4.3 is in fact an optimal one. To answer this, we build on the observation that our approach is analogous to the quicksort algorithm. This allows us to prove the following interesting proposition.

Proposition 4 *Let Δ be an arbitrary loss function. Then any comparison-based algorithm that computes the vector $\mathbf{c}$ requires $\Omega(|\mathcal{N}|\log|\mathcal{P}|)$ operations.*

Proof of Proposition 4. Since the negative samples are unsorted and the data is arbitrary, the optimal interleaving ranks may induce any possible mapping from $\{1,\ldots,|\mathcal{N}|\}$ to $\{1,\ldots,|\mathcal{P}|+1\}$. Due to Observation 2 each such mapping gives rise to a distinct possible vector $\mathbf{c}$. There are $(|\mathcal{P}|+1)^{|\mathcal{N}|}$ possibilities to be distinguished and each comparison has only two possible outcomes. Therefore we need $\log_2\left((|\mathcal{P}|+1)^{|\mathcal{N}|}\right) \in \Omega(|\mathcal{N}|\log|\mathcal{P}|)$ operations. □

Note that the above proposition only establishes an asymptotic lower bound. It does not rule out the possibility of improving the constants hidden within the asymptotic notation for a given loss function. Indeed, in the next section, we show that it is possible to exploit the additional structure of the AP loss to further speed-up our algorithm.

5 Additional Optimization for AP Loss

The computation of the optimal interleaving rank for a particular negative sample requires us to maximize the discrete function $f_j(i)$ over the domain $i \in \{1,\cdots,|\mathcal{P}|\}$ (or possibly its subdomain). Yue *et al.* [6] use a simple linear algorithm for this step, which takes $O(|\mathcal{P}|)$ time. In contrast, we propose a more efficient algorithm to maximize $\delta_j(\cdot)$, which exploits the special structure of this discrete function. This will not bring improvement in the worst-case complexity but it adds another speed-up and is interesting conceptually.

Before we describe our efficient algorithm in detail, we require the definition of a unimodal function. A discrete function $f : \{1,\cdots,p\} \leftarrow \mathbb{R}$ is said to be unimodal if and only if there exists a $k \in \{1,\cdots,p\}$ such that

$$\begin{aligned} &f(i) \leq f(i+1), \forall i \in \{1,\cdots,k-1\}, \\ &f(i-1) \geq f(i), \forall i \in \{k+1,\cdots,p\}. \end{aligned} \tag{7}$$

In other words, a unimodal discrete function is monotonically non-decreasing in the interval $[1,k]$ and monotonically non-increasing in the interval $[k,p]$. The maximization of a unimodal discrete function over its domain $\{1,\cdots,p\}$ simply requires us to find the index k that satisfies the above properties. The maximization can be performed efficiently, in $O(\log(p))$ time, using binary search.

We are now ready to state the main result that allows us to compute the optimal interleaving rank of a negative sample efficiently.

Proposition 5 *The discrete function $f_j(i)$, induced by Δ_{AP}, is unimodal in the domain $\{1,\cdots,p\}$, where $p = \min\{|\mathcal{P}|, j\}$.*

The proof of the above proposition is provided in Appendix B.

Algorithm 3: *Efficient search for the optimal interleaving rank of a negative sample.*

Input: $\{f_j(i), i = 1,\cdots,|\mathcal{P}|\}$.

1 $p = \min\{|\mathcal{P}|, j\}$

2 Compute an interleaving rank i_1 as

$$i_i = \underset{i\in\{1,\cdots,p\}}{\operatorname{argmax}} f_j(i). \tag{8}$$

3 Compute an interleaving rank i_2 as

$$i_2 = \underset{i\in\{p+1,\cdots,|\mathcal{P}|\}}{\operatorname{argmax}} f_j(i). \tag{9}$$

4 Compute the optimal interleaving rank opt_j as

$$opt_j = \begin{cases} i_1 & \text{if } f_j(i_1) \geq f_j(i_2), \\ i_2 & \text{otherwise.} \end{cases} \tag{10}$$

Using the above proposition, the discrete function $f_j(i)$ can be optimized over the domain $\{1,\cdots,|\mathcal{P}|\}$ efficiently as described in Algorithm 3. Briefly, our efficient search algorithm finds an interleaving ranking i_1 over the domain $\{1,\cdots,p\}$, where p is set to $\min\{|\mathcal{P}|, j\}$ in order to ensure that the function $f_j(\cdot)$ is unimodal (step 2 of Algorithm 3). Since i_1 can be computed using binary search, the computational complexity of this step is $O(\log(p))$. Furthermore, we find an interleaving ranking i_2 over the domain $\{p+1,\cdots,|\mathcal{P}|\}$ (step 3 of Algorithm 3). Since i_2 needs to be computed using linear search, the computational complexity of this step is $O(|\mathcal{P}|-p)$ when $p < |\mathcal{P}|$ and 0 otherwise. The optimal interleaving ranking opt_j of the negative sample $\mathbf{x}_j$ can then be computed by comparing the values of $f_j(i_1)$ and $f_j(i_2)$ (step 4 of Algorithm 3).

Note that, in a typical training dataset, the negative samples significantly outnumber the positive samples, that is, $|\mathcal{N}| \gg |\mathcal{P}|$. For all the negative samples $\mathbf{x}_j$ where $j \geq |\mathcal{P}|$,

Object class	Binary SVM	AP-SVM
Jumping	52.580	55.230
Phoning	32.090	32.630
Playing instrument	35.210	41.180
Reading	27.410	26.600
Riding bike	72.240	81.060
Running	73.090	76.850
Taking photo	21.880	25.980
Using computer	30.620	32.050
Walking	54.400	57.090
Riding horse	79.820	83.290

Table 1: *Performance of Binary* SVM *and* AP-SVM *in terms of* AP *on the test set for the different action classes of* PASCAL VOC *2011 action dataset.*

Binary SVM	AP-SVM	AP-SVM-SEARCH	AP-SVM-SELECT
0.1434	0.7154	0.0985	0.0625

Table 2: *Computation time (in seconds) for computing the most violated ranking when using the different methods. The reported time is averaged over the training for all the action classes.*

p will be equal to $|\mathcal{P}|$. Hence, the maximization of $f_j(\cdot)$ can be performed efficiently over the entire domain $\{1, \cdots, |\mathcal{P}|\}$ using binary search in $O(\log(|\mathcal{P}|))$ as opposed to the $O(|\mathcal{P}|)$ time suggested in [6].

6 Experiments

We demonstrate the efficacy of our methods, described in the previous section, on the challenging problems of action classification and object detection.

6.1 Action Classification

Dataset. We use the PASCAL VOC 2011 [20] action classification dataset for our experiments. This dataset consists of 4846 images, which include 10 different action classes. The dataset is divided into two parts: 3347 'trainval' person bounding boxes and 3363 'test' person bounding boxes. We use the 'trainval' bounding boxes for training since their ground-truth action classes are known. We evaluate the accuracy of the different instances of SSVM on the 'test' bounding boxes using the PASCAL evaluation server.

Features. We use the standard poselet [16] activation features to define the sample feature for each person bounding box. The feature vector consists of 2400 action poselet activations and 4 object detection scores. We refer the reader to [16] for details regarding the feature vector.

Methods. We show the effectiveness of our methods for both AP-SVM and NDCG-SVM. For AP-SVM, we present results for the two different efficient methods proposed in this paper. First, AP-SVM-SEARCH, which uses efficient search

Binary SVM	AP-SVM	AP-SVM-SEARCH	AP-SVM-SELECT
1.872±0.057	16.294±0.180	2.323±0.200	1.482±0.385

Table 3: *Computation time (in mili-seconds) for computing the most violated ranking per iteration when using the different methods. The reported time is averaged over all training iterations and over all the action classes.*

to compute the optimal interleaving rank for each negative sample using Algorithm 3. Second, AP-SVM-SELECT, which uses the selective ranking strategy outlined in Algorithm 2. We compare the results for our methods with that of standard binary SVM, which optimizes the 0-1 loss and also the standard AP-SVM, which uses the inefficient loss-augmented inference described in [6]. For NDCG-SVM, we present results for NDCG-SVM-SELECT, which uses the selective ranking strategy outlined in Algorithm 2. We compare the results for our method with that of standard binary SVM and also the standard NDCG-SVM, which uses the inefficient loss-augmented inference described in [8]. Note that, AP-SVM, AP-SVM-SEARCH and AP-SVM-SELECT are guaranteed to provide the same set of parameters since both efficient search and selective ranking are exact methods. Similarly, NDCG-SVM-SELECT is guaranteed to provide the same set of parameters as NDCG-SVM. The hyperparameters of all methods are fixed using 5-fold cross-validation on the 'trainval' set.

Results. Table 1 shows the AP for the rankings obtained by binary SVM and AP-SVM for the 'test' set. Note that AP-SVM (and therefore, AP-SVM-SEARCH and AP-SVM-SELECT) consistently outperforms binary SVM by optimizing a more appropriate loss function during training. The time required to compute the most violated rankings for each of the methods is shown in Table 2. Note that all the methods described in this paper result in substantial improvement in training time. The overall time required for loss-augmented inference is reduced by a factor of more than 10 compared to the original AP-SVM approach. It can also be observed that though each loss-augmented inference step for binary SVM is comparable to that for AP-SVM (Table 3), in some cases we observe that we required more cutting plane iterations for binary SVM to converge. As a result, in those cases training binary SVM is significantly slower than training AP-SVM with our proposed speed-ups.

Table 4 shows the mean NDCG for the rankings obtained by binary SVM and NDCG-SVM on the 'validation' sets when we perform 5-fold cross-validation. Note that NDCG-SVM (and therefore, NDCG-SVM-SELECT) outperforms binary SVM in most cases by optimizing a more appropriate loss function during training. The time required to compute the most violated rankings for each of the methods is shown in Table 5. The efficient method described in this paper results in substantial improvement in training time. The overall time required for loss-augmented inference is reduced by

a factor of more than 100 compared to the original NDCG-SVM approach. It can also be observed that loss-augmented inference step for our version of NDCG-SVM optimization is significantly faster than that of binary SVM.

In order to understand the effect of the size and composition of the dataset on our approaches, we perform 3 experiments with variable number samples for the action class phoning. First, we vary the total number of samples while having a constant positive to negative ratio of $1:10$. Second, we vary the number of negative samples while fixing the number of positive samples to 227. Third, we vary the number of positive samples while fixing the number of negative samples to 200. As can be seen in Fig. 1 and Fig. 2, the time required to perform loss-augmented inference is significantly lower for our methods for both AP-SVM and NDCG-SVM.

Object class	Binary SVM	NDCG-SVM
Jumping	86.409	87.895
Phoning	73.134	76.733
Playing instrument	81.533	83.666
Reading	74.528	75.588
Riding bike	94.928	95.958
Running	93.766	93.776
Taking photo	74.058	76.701
Using computer	79.518	78.276
Walking	89.789	89.742
Riding horse	96.160	96.875

Table 4: *Performance of Binary* SVM *and* NDCG-SVM *in terms of* NDCG *on the validation set for the different action classes of* PASCAL VOC *2011 action dataset. We conduct 5-fold cross-validation and report the mean* NDCG *over the five validation sets.*

Binary SVM	NDCG-SVM	NDCG-SVM-SELECT
0.1687	6.8019	0.0473

Table 5: *Computation time (in seconds) for computing the most violated ranking when using the different methods. The reported time is averaged over the training for all the action classes.*

6.2 Object Detection

Dataset. We use the PASCAL VOC 2007 [20] object detection dataset, which consists of a total of 9963 images. The dataset is divided into a 'trainval' set of 5011 images and a 'test' set of 4952 images. All the images are labelled to indicate the presence or absence of the instances of 20 different object categories. In addition, we are also provided with tight bounding boxes around the object instances, which we ignore during training and testing. Instead, we treat the location of the objects as a latent variable. In order to reduce the latent variable space, we use the selective-search algorithm [21] in its fast mode, which generates an average of 2000 candidate windows per image.

Binary SVM	NDCG-SVM	NDCG-SVM-SELECT
2.447±0.259	71.066±1.569	0.548±0.111

Table 6: *Computation time (in mili-seconds) for computing the most violated ranking per iteration when using the different methods. The reported time is averaged over all training iterations and over all the action classes.*

Features. For each of the candidate windows, we use a feature representation that is extracted from a trained Convolutional Neural Network (CNN). Specifically, we pass the image as input to the CNN and use the activation vector of the penultimate layer of the CNN as the feature vector. Inspired by the work of Girshick *et al.* [22], we use the CNN that is trained on the ImageNet dataset [23], by rescaling each candidate window to a fixed size of 224×224. The length of the resulting feature vector is 4096.

Methods. We train latent AP-SVMs [24] as object detectors for 20 object categories. In our experiments, we determine the value of the hyperparameters using 5-fold cross-validation. During testing, we evaluate each candidate window generated by selective search, and use non-maxima suppression to prune highly overlapping detections.

Results. This experiment places high computational demands due to the size of the dataset (5011 'trainval' images), as well as the size of the latent space (2000 candidate windows per image). We compare the computational efficiency of the loss-augmented inference algorithm proposed in [6] and the exact methods proposed by us. The total time taken for loss-augmented inference during training, averaged over the all the 20 classes, is 0.5214 sec for our efficient exact methods (SEARCH+SELECT) which is significantly better than the 7.623 sec taken by the algorithm used in [6].

7 Discussion

We provided a characterization of ranking based loss functions that are amenable to a quicksort based optimization algorithm for obtaining the corresponding most violated constraint. We proved that the quicksort flavored algorithm provides a better computational complexity than the state of the art methods for AP and NDCG loss functions. Furthermore, we established that the complexity of our algorithm cannot be improved upon asymptotically by any comparison based method. We empirically demonstrated the efficacy of our approach on two challenging real world problems—action classification and object detection—using standard publicly available datasets.

In theory, our approach can readily be used in conjunction with other learning frameworks, such as the popular deep convolutional neural networks. A combination of methods

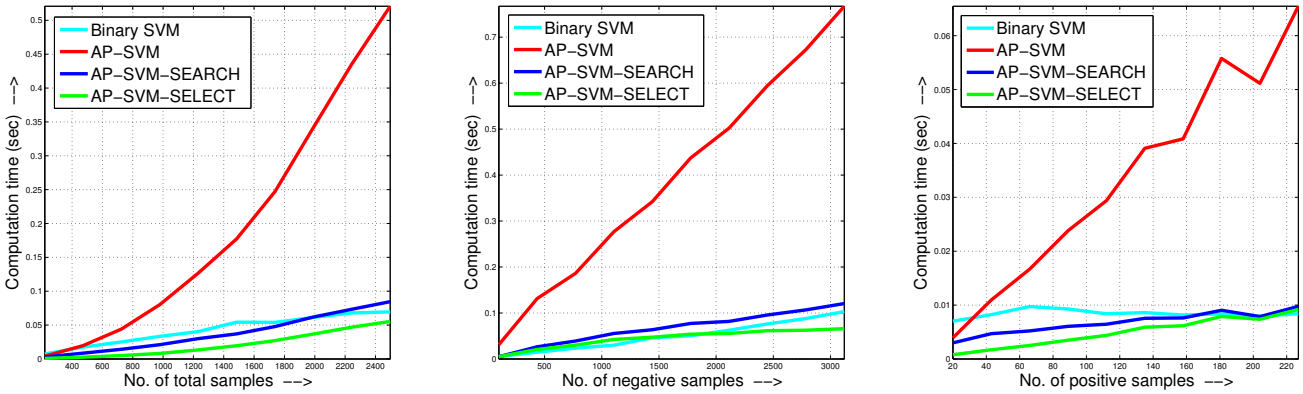

Figure 1: *Computation time for solving all the loss augmented inference problems during the complete training of the Binary* SVM*s and* AP-SVM*s, while the no. of total, negative and positive samples are varied.*

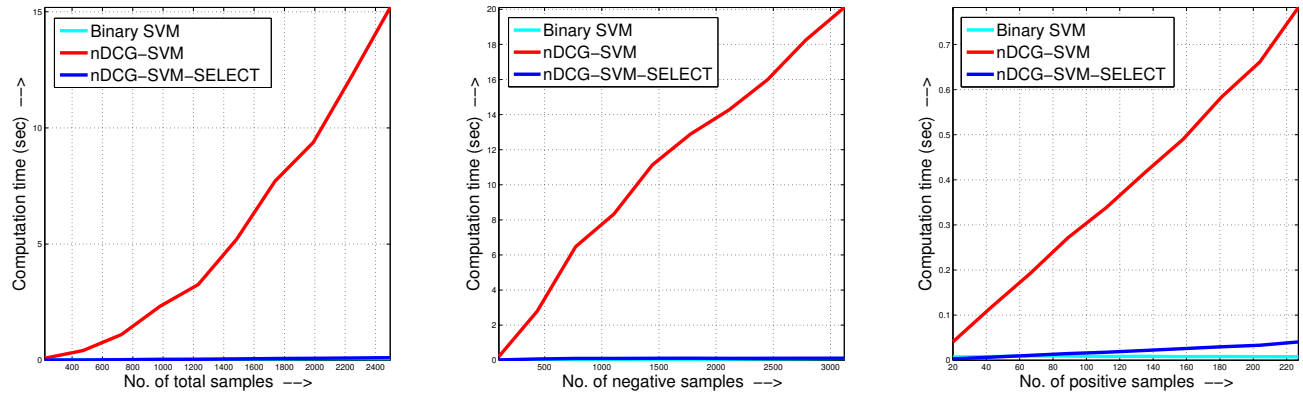

Figure 2: *Computation time for solving all the loss augmented inference problems during the complete training of the Binary* SVM*s and* NDCG-SVM*s, while the no. of total, negative and positive samples are varied.*

proposed in this paper and the speed-ups proposed in [25] may prove to be effective in such a framework. The efficacy of optimizing ranking based loss functions efficiently using other frameworks needs to be empirically evaluated. Another computational bottleneck of all rank SVM frameworks is the computation of the joint feature vector. An interesting direction of future research would be to combine our approaches with those of sparse feature coding [26, 27, 28] to improve the speed to rank SVM learning further.

A Proofs of observations from Section 4

In this part of the appendix we will prove Observations 1-4 from Section 4.

Lemma 6 *The joint feature vector* $\Psi(\mathbf{X},\mathbf{R})$ *decomposes into contributions of negative and positive samples as follows:*

$$\begin{aligned}\Psi(\mathbf{X},\mathbf{R}) &= \frac{1}{|\mathcal{P}|\,|N|}\sum_{\mathbf{x}\in\mathcal{P}}\sum_{\mathbf{y}\in\mathcal{N}}\mathbf{R}_{\mathbf{x},\mathbf{y}}(\psi(\mathbf{x})-\psi(\mathbf{y}))\\ &= \sum_{\mathbf{x}\in\mathcal{P}} c(\mathbf{x})\psi(\mathbf{x}) + \sum_{\mathbf{y}\in\mathcal{N}} c(\mathbf{y})\psi(\mathbf{y}),\end{aligned}$$

where

$$c(\mathbf{x}) = \frac{|\mathcal{P}|+2-2rank(\mathbf{x})}{|\mathcal{P}|\,|\mathcal{N}|},\; c(\mathbf{y}) = \frac{|\mathcal{N}|+2-2rank(\mathbf{y})}{|\mathcal{P}|\,|\mathcal{N}|}.$$

In particular, assuming already that $\{s_i^+\}$ *is sorted, and that* $\mathbf{R}$ *is induced by a vector* $\mathbf{r}\in\mathcal{R}$*, one has*

$$\mathbf{w}^T\Psi(\mathbf{X},\mathbf{R}) = \sum_{i=1}^{|\mathcal{P}|} c_i^+ s_i^+ + \sum_{j=1}^{|\mathcal{N}|} c_j^- s_j^*,$$

where

$$c_i^+ = \frac{|\mathcal{P}|+2-2r_i^+}{|\mathcal{P}|\,|\mathcal{N}|}, \qquad c_j^- = \frac{|\mathcal{N}|+2-2r_j}{|\mathcal{P}|\,|\mathcal{N}|}.$$

Here r_i^+ *stands for the interleaving rank of the* i*-th positive sample, which can be computed as* $r_i^+ = 1+|\{j : r_j \le i\}|$.

Observation 1 *There exists an optimal solution* $\hat{\mathbf{R}}$ *of* (5) *in which the positive samples appear in the descending order of their scores* s_i *and also the negative samples appear in descending order of their scores.*

Proof. Let $\mathbf{R}$ be any optimal solution and let $\hat{\mathbf{R}}$ be a ranking with the same $\pm$-pattern as $\mathbf{R}$ but both negative and positive samples sorted decreasingly by their scores. Then due to (C1) we have $\Delta(\mathbf{R}^*, \mathbf{R}) = \Delta(\mathbf{R}^*, \hat{\mathbf{R}})$. It remains to check that $\mathbf{w}^T\Psi(\mathbf{X}, \mathbf{R})$ does not decrease if we swap in $\mathbf{R}$ two samples $x, y \in \mathcal{P}$ with $ind(\mathbf{x}) < ind(\mathbf{y})$ and $s(\mathbf{x}) < s(\mathbf{y})$ (it boils down to $ac+bd \geq ad+bc$ for $a \geq b \geq 0$ and $c \geq d \geq 0$). Since similar argument applies for negative samples one can perform swaps in $\mathbf{R}$ until reaching $\hat{\mathbf{R}}$ without decreasing the value of the objective. □

Observation 2 *Assume we are given a vector* $\mathbf{r} \in \mathbf{R}$ *and an array* $\{s_j^{**}\}$ *which is a rearrangement of the array* $\{s_j^-\}$ *in which* $s_i^{**} > s_j^{**}$ *whenever* $r_i < r_j$. *Then we can compute the entire output of the algorithm, that is the vector* $\mathbf{c}$ *as well as the value* $\Delta(\mathbf{R}^*, \mathbf{R_r})$ *in time* $O(|\mathcal{N}| + |\mathcal{P}|)$. *Moreover, the mapping that produces vector* $\mathbf{c}$ *given the vector* $\mathbf{r}$ *is injective.*

Proof. Note that the interleaving ranks for negative samples determine the $\pm$-pattern of the sought ranking $\mathbf{R_r}$. Moreover this $\pm$-pattern can be constructed in one pass over $\{r_j\}$ and by (C1) the loss $\Delta(\mathbf{R}^*, \mathbf{R_r})$ can be computed from it, all that in $O(|\mathcal{N}| + |\mathcal{P}|)$ time. It remains to find the coefficients $\{c_i^+\}$ and $\{c_i^-\}$. But since $c_i^- = c_j^-$ whenever $r_i = r_j$ the coefficient c_i^- (of the score s_i^*) can be also associated with the score $\{s_i^{**}\}$. Computing $\{c_i^+\}$ from $\{c_i^-\}$ takes also $O(|\mathcal{P}| + |\mathcal{N}|)$ time by Lemma 6. It is also clear that already the coefficients $\{c_i^-\}$ are uniquely determined by $\mathbf{r}$ hence the mapping is indeed injective. □

Observation 3 *There are functions* $f_j \colon \{1, \ldots, |\mathcal{P}|+1\} \to \mathbb{R}$ *for* $j = 1, \ldots, |\mathcal{N}|$ *such that for a proper ranking* $\mathbf{R}$ *which corresponds to some vector* $\mathbf{r} \in \mathcal{R}$ *the objective function* (5) *can be written as*

$$\sum_{j=1}^{|\mathcal{N}|} f_j(r_j),$$

where the functions f_j *inherit property (C3) and provided the value of* s_j^* *also the property (C4). In particular, given* s_j^* *we can compute all the values of* $f_j(i)$ *for* $l \leq i \leq r$ *in* $O(r-l)$ *time.*

Proof. We can use that the array $\{s_i^+\}$ is sorted to write the decomposition from Lemma 6 also as

$$\begin{aligned}
&\mathbf{w}^T\Psi(\mathbf{X}, \mathbf{R}) \\
&= \frac{1}{|\mathcal{N}|\,|\mathcal{P}|}\mathbf{w}^T \sum_{\mathbf{y}\in\mathcal{N}} \left(c(\mathbf{y})\psi(\mathbf{y}) + \sum_{\mathbf{x}\in\mathcal{P}} \mathbf{R}_{\mathbf{x},\mathbf{y}}\psi(\mathbf{x}) \right) \\
&= \frac{1}{|\mathcal{N}|\,|\mathcal{P}|} \sum_{j=1}^{|\mathcal{N}|} \left((|\mathcal{P}|+2-2r_j)s_j^* + 2\sum_{i=1}^{r_j-1} s_i^+ - \sum_{i=1}^{|\mathcal{P}|} s_i^+ \right).
\end{aligned}$$

This, in combination with (C2), defines the functions f_j for $j = 1, \ldots, |\mathcal{N}|$. As for the conditions (C3) and (C4), we have

$$f_j(i+1) - f_j(i) = \frac{2(s_i^+ - s_j^*)}{|\mathcal{N}|\,|\mathcal{P}|}.$$

Now (C3) reduces to the obvious $s_{j+1}^* \leq s_j^*$ and indeed knowing s_j^* implies constant-time evaluation. □

Observation 4 (Correctness of greedy approach) *If* $i < j$, *then* $opt_i \leq opt_j$. *In other words* $\mathbf{opt} \in \mathcal{R}$.

Proof. Recall that opt_j is the highest rank with maximal value of the corresponding $f_{j'}$. It suffices to prove that for $i_{j+1} = \max \operatorname{argmax} f_{j+1}$ and $i_j = \max \operatorname{argmax} f_j$, we have $i_{j+1} \geq i_j$. Since by Observation 3 we can compare the discrete derivatives of f_j and f_{j+1}, all left to do is to formalize the discrete analogue of what seems intuitive for continuous functions.

Assume $i_{j+1} < i_j$. Then since

$$\begin{aligned}
f_{j+1}(i_j) - f_{j+1}(i_{j+1}) &= \sum_{i=i_{j+1}}^{i_j-1} f_{j+1}(i+1) - f_{j+1}(i) \\
&\geq \sum_{i=i_{j+1}}^{i_j-1} f_j(i+1) - f_j(i) \\
&= f_j(i_j) - f_j(i_{j+1}) \geq 0,
\end{aligned}$$

we obtain that $i_j \in \operatorname{argmax} f_{j+1}$ and as $i_j > i_{j+1} = \max \operatorname{argmax} f_{j+1}$ and we reached the expected contradiction. □

Remark 1 *Observation 4 is not true for* Δ_{NDCG} *with function* $D(i)$ *taken from [8] as*

$$D(i) = \begin{cases} 1 & 1 \leq i \leq 2 \\ 1/\log_2(i) & i > 2 \end{cases}.$$

Proof. Consider negative samples $\mathbf{x}_1$ and $\mathbf{x}_2$ and a positive sample $\mathbf{x}_3$ with scores $s_1 = 3\varepsilon$, $s_2 = \varepsilon$, $s_3 = 5\varepsilon$, where $\varepsilon > 0$ is small.

Note that the NDCG loss of a ranking $\mathbf{R}$ reduces to $\Delta_{NDCG}(\mathbf{R}^*, \mathbf{R}) = 1 - D(ind(\mathbf{x}_3))$ where we used the fact that $D(1) = 1$.

The decomposition $\Delta_{NDCG}(\mathbf{R}^*, \mathbf{R}) = \delta_1(r_1) + \delta_2(r_2)$ holds if we set

$$\begin{aligned}
\delta_1(1) &= \delta_2(1) = 0, \\
\delta_2(1) &= D(2) - D(3), \\
\delta_1(2) &= D(1) - D(2) = 0
\end{aligned}$$

and (possibly by looking at the proof of Observation 3) we also find values of f_1 and f_2 as

$$\begin{aligned} f_1(1) &= \frac{1}{2}(s_1 - s_3) + \delta_1(1) = -\varepsilon < \varepsilon \\ &= \frac{1}{2}(s_3 - s_1) + \delta_1(2) = f_1(2) \\ f_2(1) &= \frac{1}{2}(s_2 - s_3) + \delta_2(1) = -2\varepsilon + D(2) - D(3) > 2\varepsilon \\ &= \frac{1}{2}(s_3 - s_2) + \delta_2(2) = f_2(2). \end{aligned}$$

Hence $opt_1 = 2 > 1 = opt_2$, a contradiction. □

B Properties of Δ_{AP} and Δ_{NDCG}

In this place, let us prove the aforementioned properties of Δ_{AP} and Δ_{NDCG}.

Proposition 7 *Δ_{NDCG} is QS-suitable.*

Proof. The property (C1) is immediate. For the others, let us first verify that the functions δ_j can be set as

$$\delta_j(i) = \frac{1}{C}\left(D(i+j-1) - D(|\mathcal{P}|+j)\right),$$

where $C = \sum_{i=1}^{|\mathcal{P}|} D(i)$. Indeed, one can check that

$$\begin{aligned} &\Delta(\mathbf{R}^*, \mathbf{R}) \\ &= 1 - \frac{\sum_{\mathbf{x}\in\mathcal{P}} D(ind(\mathbf{x}))}{\sum_{i=1}^{|\mathcal{P}|} D(i)} \\ &= \frac{1}{C}\sum_{i=1}^{|\mathcal{P}|} D(i) - \sum_{\mathbf{x}\in\mathcal{P}} D(ind^+(\mathbf{x}) + rank(\mathbf{x}) - 1) \\ &= \frac{1}{C}\sum_{\mathbf{x}\in\mathcal{N}} D(ind^-(\mathbf{x}) + rank(\mathbf{x}) - 1) - D(|\mathcal{P}| + ind^-(\mathbf{x})) \\ &= \sum_{\mathbf{x}\in\mathcal{N}} \delta_{ind^-(\mathbf{x})}(rank(\mathbf{x})) \end{aligned}$$

as desired. As for (C3) and (C4), let us realize that

$$\delta_j(i+1) - \delta_j(i) = \frac{1}{C}\left(D(i+j) - D(i+j-1)\right).$$

Then (C4) becomes trivial and checking (C3) reduces to

$$D(i+j+1) + D(i+j-1) \geq 2D(i+j)$$

which follows from convexity of the function D. □

Proposition 8 *Δ_{AP} is QS-suitable.*

Proof. Again, the property (C1) goes without saying. The functions δ_j were already identified in [6] as

$$\delta_j(i) = \frac{1}{|\mathcal{P}|}\sum_{k=i}^{|\mathcal{P}|}\left(\frac{j}{j+k} - \frac{j-1}{j+k-1}\right)$$

so after writing

$$\delta_j(i+1) - \delta_j(i) = \frac{j-1}{j+i-1} - \frac{j}{j+i}$$

we again have (C4) for free and (C3) reduces to

$$2g_i(j) \geq g_i(j-1) + g_i(j+1),$$

where $g_i(x) = \frac{x}{x+i}$, and the conclusion follows from concavity of $g_i(x)$ for $x > 0$. □

In the rest of this section we focus on proving the unimodality stated in Proposition 5.

Proposition 5 (Restated) *The discrete function $f_j(i)$, induced by Δ_{AP}, is unimodal in the domain $\{1, \cdots, p\}$, where $p = \min\{|\mathcal{P}|, j\}$.*

Let us first state the following lemmas, which easily lead to the proposition. Knowing the form of $\delta_j(i)$ for Δ_{AP} from previous paragraphs, we may split the summand term in the summation $f_j(i)$ as follows:

$$f_j(i) = \sum_{k=i}^{|\mathcal{P}|} g_1(j,k) + \sum_{k=i}^{|\mathcal{P}|} g_2(j,k) - C_j,$$

where

$$\begin{aligned} g_1(j,k) &= \frac{1}{|\mathcal{P}|}\left(\frac{j}{j+k} - \frac{j-1}{j+k-1}\right) \\ g_2(j,k) &= -\frac{2\left(s_k^+ - s_j^*\right)}{|\mathcal{P}|\,|\mathcal{N}|} \\ C_j &= \sum_{k=1}^{|\mathcal{P}|} g_2(j,k) \end{aligned}$$

Lemma 6 *For $k < j$, $g_1(j,k)$ monotonically decreases with decreasing k, that is $\forall\ k < j\ g_1(j,k-1) \leq g_1(j,k)$.*

Proof. For $j \geq 1$ and $k \geq 1$, $(j+k) > j \Rightarrow j(j+k) - (j+k) < j(j+k) - j \Rightarrow \frac{j}{j+k} > \frac{j-1}{j+k-1}$. So, term $g_1(j,k) > 0$ for all $k \geq 1$. It can also be verified that the function $g_1(j,k)$ is 0 at 0 and has a single maxima for $k \in \Re^+$, at $k = \sqrt{j(j-1)}$. From this we can conclude that for discrete $k \in \mathbb{Z}^+$, $g_1(j,k)$ would have maximum value either at $k = j$ or $k = j-1$. Therefore, for $k < j$, $g_1(j,k)$ would monotonically decrease with decreasing k. □

Lemma 7 *For $k < j$, $g_2(j,k)$ monotonically decreases with decreasing k, that is $\forall\ k < j\ g_2(j,k-1) \leq g_2(j,k)$.*

Proof. In $g_2(j,k)$, the negative score s_j^* is a constant for a given j. Whereas, the positive scores s_k^+ being sorted in descending order, monotonically increase as k decreases. Therefore, $g_2(j,k)$ which is $-s_k^+ + constant$, monotonically decreases as k decreases. □

Proof of Proposition 5: From Lemmas 6 and 7, for $k < j$, $g_1(j,k)$ and $g_2(j,k)$ monotonically decreases with decreasing k. As a result, $g_1(j,k) + g_2(j,k)$ also monotonically decreases when k is decreased from right to left of the number line. Here, there can be 3 scenarios,

(i) $(g_1(j,1) + g_2(j,1)) \geq 0$. In this case, as the function is monotonic and decreases towards left,

$$\begin{aligned} & (g_1(j,i) + g_2(j,i)) \geq 0, for\ i \in \{1,2,...,j\} \\ \Rightarrow \quad & f_j(i) - f_j(i+1) \geq 0, for\ \ i \in \{1,2,...,\} \\ \Rightarrow \quad & f_j(i) \geq f_j(i+1), for\ \ i \in \{1,2,...,\} \end{aligned}$$

Therefore, according to definition of unimodality, $f_j(i)$ would be unimodal with $k = 1$.

(ii) $(g_1(j,j-1) + g_2(j,j-1)) \leq 0$. In this case, using similar reasoning as above,

$$\begin{aligned} & (g_1(j,i) + g_2(j,i)) \leq 0, for\ i \in \{j-1,...,1\} \\ \Rightarrow \quad & f_j(i) - f_j(i+1) \leq 0, for\ \ i \in \{j-1,...,1\} \\ \Rightarrow \quad & f_j(i) \leq f_j(i+1), for\ \ i \in \{j-1,...,1\} \end{aligned}$$

Therefore, $f_j(i)$ would be unimodal with $k = j - 1$.

(iii) $(g_1(j,1) + g_2(j,1)) \leq 0$ and at the same time $(g_1(j,j-1) + g_2(j,j-1)) \geq 0$. In this case, there should exist a point across which the function $(g_1 + g_2)$ changes its sign from positive to negative when moving from right to left. In other words, there should exist $k \in 1, 2, \ldots, j-1$, such that,

$$\begin{aligned} & (g_1(j,i) + g_2(j,i)) \geq 0, i \in \{k+1,...,j\} \\ & (g_1(j,i) + g_2(j,i)) \leq 0, i \in \{1,...,k\} \\ \Rightarrow \quad & f_j(i) - f_j(i+1) \geq 0, for\ \ i \in \{k,...,j-1\} \\ & f_j(i) - f_j(i+1) \leq 0, for\ \ i \in \{j-1,...,1\} \\ \Rightarrow \quad & f_j(i) \geq f_j(i+1), for\ \ i \in \{k,...,j-1\} \\ & f_j(i) \leq f_j(i+1), for\ \ i \in \{j-1,...,1\} \end{aligned}$$

Here too, $f_j(i)$ satisfies the conditions for unimodality with k being the maximum point.

In all the 3 of the exhaustive cases, $f_j(i)$ satisfies the conditions for unimodality. Hence, $f_j(i)$ is unimodal in the region $\{1, 2, \ldots, j-1\}$. As a function which is unimodal in a certain region would also be unimodal in a subset of the region, $f_j(i)$ is unimodal in the region $\{1, 2, \ldots, p\}$, where, $p = \min(|\mathcal{P}|, j)$. □

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